3.4 REGLA DE L'HÔPITAL

REGLA DE L'HÔPITAL

Sean f y g funciones derivables en un entorno $(a-r;\ a+r)$ del punto a.

Si $\lim\limits_{x\to a} f(x) = 0$, $\lim\limits_{x\to a} g(x) = 0$, y existe $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$, entonces también existe $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$

y es: $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

EJERCICIO RESUELTO

Calcula estos límites, aplicando la regla de L'Hôpital:

a) $\lim\limits_{x\to 1} \dfrac{2x^2+2x-4}{x^3-4x^2-3x}$ **b)** $\lim\limits_{x\to 0} \dfrac{e^x-1}{sen\ x}$ **c)** $\lim\limits_{x\to 0} \dfrac{x^2 e^x}{x^3-3x^2}$

RESOLUCIÓN

a) $\lim\limits_{x\to 1} \dfrac{2x^2+2x-4}{x^3-4x^2+3x} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 1} \dfrac{4x+2}{3x^2-8x+3} = \dfrac{6}{-2} = -3$

b) $\lim\limits_{x\to 0} \dfrac{e^x-1}{sen\ x} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{e^x}{\cos x} = \dfrac{1}{1} = 1$

c) $\lim\limits_{x\to 0} \dfrac{x^2 e^x}{x^3-3x^2} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{(2x+x^2)\,e^x}{3x^2-6x} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{(2+4x+x^2)\,e^x}{6x-6} = \dfrac{-1}{3}$

1 Calcula los siguientes límites, aplicando la regla de L'Hôpital:

a) $\lim\limits_{x\to 0} \dfrac{x^5-3x^4+2x}{x^2-3x}$ b) $\lim\limits_{x\to 0} \dfrac{2^x-1}{x^2}$ c) $\lim\limits_{x\to 0} \dfrac{\ln(x+1)}{x}$

d) $\lim\limits_{x\to 0} \dfrac{x+sen\ x}{x}$ e) $\lim\limits_{x\to 1} \dfrac{3^x-3}{\ln x}$ f) $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{2x^2}$

g) $\lim\limits_{x\to 0} \dfrac{x+tg\ x}{x+sen\ x}$ h) $\lim\limits_{x\to 0} \dfrac{arc\ tg\ x - x + x^3/3}{x^3}$ i) $\lim\limits_{x\to 2} \dfrac{e^x-e^2}{\ln(x^2-3)}$

1 a) $\dfrac{-2}{3}$

b) $\lim\limits_{x\to 0^-} f(x) = -\infty$; $\lim\limits_{x\to 0^+} f(x) = +\infty$

c) 1

d) 2

e) $3\ \ln 3$

f) $\dfrac{1}{2}$

g) 1

h) 0

i) $e^2/4$

AMPLIACIÓN DE LA REGLA DE L'HÔPITAL

- Los límites del tipo $\lim_{x \to a} \frac{f(x)}{g(x)}$, donde a es $-\infty$, $+\infty$ o un número, si dan lugar a una indeterminación del tipo $\left(\frac{0}{0}\right)$ o $\left(\frac{\pm\infty}{\pm\infty}\right)$, pueden obtenerse derivando numerador y denominador y calculando (si existe) el límite del cociente de sus derivadas.
- Hay expresiones del tipo $(\infty - \infty)$, (1^{∞}) u otras que, con un poco de habilidad, se pueden poner en forma de cociente para que se pueda aplicar la relga de L'Hôpital.

EJERCICIO RESUELTO

Calcula los siguientes límites, aplicando la regla de L'Hôpital:

a) $\lim_{x \to +\infty} \frac{\ln (x+1)}{x+3}$ ***b)*** $\lim_{x \to +\infty} x\,(e^{1/x} - 1)$ ***c)*** $\lim_{x \to 0} (1 + \text{sen}\, x)^{1/x}$

RESOLUCIÓN

a) $\lim_{x \to +\infty} \frac{\ln (x+1)}{x+3} = \left(\frac{\infty}{\infty}\right) = \lim_{x \to +\infty} \frac{\frac{1}{x+1}}{1} = 0$

b) $\lim_{x \to +\infty} x\,(e^{1/x} - 1) = (+\infty \cdot 0) = \lim_{x \to +\infty} \frac{e^{1/x} - 1}{1/x} = \left(\frac{0}{0}\right) = \lim_{x \to \infty} \frac{(-1/x^2)\, e^{1/x}}{-1/x^2} = 1$

1^{∞} c) $\lim_{x \to 0} (1 + \text{sen}x)^{1/x} \Rightarrow \lim_{x \to 0} (\ln (1 + \text{sen}x)^{1/x}) = \lim_{x \to 0} \left(\frac{1}{x} \cdot \ln (1 + \text{sen}x)\right) = \lim_{x \to 0} \frac{\ln (1 + \text{sen}x)}{x} =$

$= \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\frac{\cos x}{1 + \text{sen}x}}{1} = 1 \Rightarrow \lim_{x \to 0} (1 + \text{sen}x)^{1/x} = e^1 = e.$

2 Calcula estos límites:

a) $\lim_{x \to 0} (3x)^{-1} \cdot \text{sen}\, x$

b) $\lim_{x \to +\infty} x^2 (1 - e^{1/x^2})$ $+\infty \cdot 0$

c) $\lim_{x \to +\infty} \frac{\sqrt{x+1}}{e^x}$

d)= $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$ $\infty - \infty$

e) $\lim_{x \to 0} \left(\frac{2}{x} - \frac{2}{\text{sen}\, x}\right)$ $\infty - \infty$

f) $\lim_{x \to 0} (1 - \text{sen}\, x)^{1/x}$ 1^{∞}

g)= $\lim_{x \to 0^+} \left(\ln \frac{1}{x}\right)^x$ ∞^0

h) $\lim_{x \to -1} (x+1)^{x+1}$ (0^0)

i) $\lim_{x \to \pi/2} (\text{sen}\, x)^{\text{tg}\, x}$

2 a) $\frac{1}{3}$

b) -1

c) 0

d) $\frac{1}{2}$

e) 0

f) e^{-1}

g) 1

h) 1

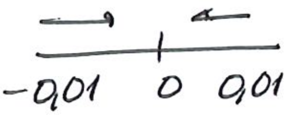
i) 1

L'HOPITAL (FITXA)

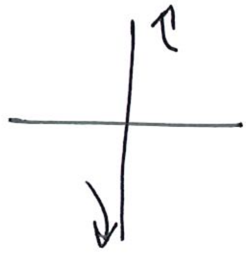
40. on → $\boxed{1}$

a) $\lim\limits_{x\to 0} \dfrac{x^5-3x^4+2x}{x^2-3x} = \left(\dfrac{0}{0}\right) \overset{H}{=} \lim\limits_{x\to 0} \dfrac{5x^4-12x^3+2}{2x-3} = \boxed{\dfrac{-2}{3}}$

b) $\lim\limits_{x\to 0} \dfrac{2^x-1}{x^2} = \left(\dfrac{0}{0}\right) \overset{H}{=} \lim\limits_{x\to 0} \dfrac{2^x\cdot\ln 2}{2x} = \left(\dfrac{\ln 2}{0}\right)$ IND.



$$\left\{\begin{array}{l} \lim\limits_{x\to 0^-} \dfrac{2^x\ln 2}{2x} = \dfrac{\ln 2}{0^-} = -\infty \\ \lim\limits_{x\to 0^+} \dfrac{2^x\ln 2}{2x} = \dfrac{\ln 2}{0^+} = +\infty \end{array}\right.$$



c) $\lim\limits_{x\to 0} \dfrac{\ln(x+1)}{x} = \dfrac{\ln 1}{0} = \left(\dfrac{0}{0}\right) \overset{H}{=} \lim\limits_{x\to 0} \dfrac{\frac{1}{x+1}}{1} = \lim\limits_{x\to 0} \dfrac{1}{x+1} = \boxed{1}$

d) $\lim\limits_{x\to 0} \dfrac{x+\sin x}{x} = \dfrac{0+\sin 0}{0} = \left(\dfrac{0}{0}\right) \overset{H}{=} \lim\limits_{x\to 0} \dfrac{1+\cos x}{1} = 1+1 = 2$

e) $\lim\limits_{x\to 1} \dfrac{3^x-3}{\ln x} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 1} \dfrac{3^x\cdot\ln 3}{1/x} = \dfrac{3\ln 3}{1/1} = \boxed{3\ln 3}$

f) $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{2x^2} = \dfrac{1-1^2}{0} = \left(\dfrac{0}{0}\right) \overset{H}{=} \lim\limits_{x\to 0} \dfrac{-2\cos x\cdot(-\sin x)}{2\cdot 2x}$

$= \lim\limits_{x\to 0} \dfrac{2\sin x\cos x}{4x} = \dfrac{2\overset{0}{\cancel{\sin 0}}\cdot\overset{1}{\cancel{\cos 0}}}{4\cdot 0} = \left(\dfrac{0}{0}\right) =$

$\overset{H}{=} \lim\limits_{x\to 0} \dfrac{\cos x\cdot\cos x + \sin x(-\sin x)}{2} = \lim\limits_{x\to 0} \dfrac{\cos^2 x - \sin^2 x}{2}$

$= \dfrac{1^2 - 0}{2} = \boxed{\dfrac{1}{2}}$

g) $\lim\limits_{x\to 0} \dfrac{x+\operatorname{tg} x}{x+\sin x} = \left(\dfrac{0}{0}\right) \overset{H}{=} \lim\limits_{x\to 0} \dfrac{1+\frac{1}{\cos^2 x}}{1+\cos x} = \dfrac{1+\frac{1}{1}}{1+1} = 1$

h) $\lim_{x\to 0} \frac{\operatorname{arctg} x - x + \frac{x^3}{3}}{x^3} = \frac{0-0+0}{0} = \left(\frac{0}{0}\right)$

$\overset{H}{=} \lim_{x\to 0} \frac{\frac{1}{1+x^2} - 1 + \frac{\cancel{3}x^2}{\cancel{3}}}{3x^2} = \frac{1-1+0}{0} = \left(\frac{0}{0}\right) =$

$= \lim_{x\to 0} \frac{1-(1+x^2)+x^2(1+x^2)}{(1+x^2)\cdot 3x^2} = \lim_{x\to 0} \frac{\cancel{1}\cancel{-1}\cancel{-x^2}\cancel{+x^2}+x^4}{(1+x^2)3x^2}$

$\lim_{x\to 0} \frac{x^4}{3x^2+3x^4} = \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{x^4}{x^2(3+3x^2)} =$

$= \lim_{x\to 0} \frac{x^2}{3+3x^2} = \frac{0}{3} = 0$

I) $\lim_{x\to 2} \frac{e^x - e^2}{\ln(x^2-3)} = \frac{e^2-e^2}{\ln(4-3)} = \frac{0}{\ln 1} = \left(\frac{0}{0}\right)$

$= \lim_{x\to 2} \frac{e^x}{\frac{1}{x^2-3}\cdot 2x} = \lim_{x\to 2} \frac{e^x(x^2-3)}{2x} = \frac{e^2(4-3)}{2\cdot 2} = \boxed{\frac{e^2}{4}}$

L'HOPITAL

$\boxed{2}$ a) $\lim_{x\to 0} (3x)^{-1}\cdot \sin x = \lim_{x\to 0} \frac{\sin x}{3x} = \left(\frac{0}{0}\right) \overset{H}{=}$

$= \lim_{x\to 0} \frac{\cos x}{3} = \boxed{\frac{1}{3}}$

b) $\lim_{x\to +\infty} x^2(1-e^{1/x^2}) = +\infty(1-e^{1/\infty}) = +\infty \cdot 0.$

$= \lim_{x\to +\infty} \frac{1-e^{1/x^2}}{1/x^2} = \frac{0}{1/\infty} = \left(\frac{0}{0}\right) \overset{H}{=} \lim_{x\to +\infty} \frac{-e^{1/x^2}\cdot \cancel{(-2)} \cancel{x^{-3}}}{\cancel{-2}\cdot \cancel{x^{-3}}} =$

$= \lim_{x\to +\infty} -e^{1/x^2} = -e^{1/\infty} = -e^0 = \boxed{-1}$

c) $\lim_{x\to +\infty} \frac{\sqrt{x+1}}{e^x} = \left(\frac{+\infty}{+\infty}\right) \overset{H}{=} \lim_{x\to +\infty} \frac{\frac{1}{2\sqrt{x+1}}}{e^x} = \frac{1}{+\infty} = \boxed{0}$

d) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \left(\frac{1}{0}\right) - \left(\frac{1}{0}\right)$ IND. $= (\pm\infty - \pm\infty)$

$= \lim_{x\to 1} \frac{(x-1)\cdot - \ln x}{\ln x\,(x-1)} = \left(\frac{0}{0}\right) \overset{H}{=} \lim_{x\to 1} \frac{1 - 1/x}{\frac{1}{x}(x-1) + \ln x} =$

$= \lim_{x\to 1} \frac{\frac{x-1}{x}}{\frac{x-1+x\cdot\ln x}{x}} = \lim_{x\to 1} \frac{x-1}{x-1+x\ln x} = \left(\frac{0}{0}\right) \overset{H}{=}$

$\overset{H}{=} \lim_{x\to 1} \frac{1}{1+(\ln x + x/x)} = \lim_{x\to 1} \frac{1}{1+\ln x+1} = \lim_{x\to 1} \frac{1}{2+\ln x} = \boxed{\frac{1}{2}}$

e) $\lim_{x\to 0} \left(\frac{2}{x} - \frac{2}{\sin x}\right) = \left(\frac{2}{0}\right) - \left(\frac{2}{0}\right) = (\pm\infty - (\pm\infty))$

$\lim_{x\to 0} \frac{2\sin x - 2x}{x\cdot\sin x} = \left(\frac{0}{0}\right) \overset{H}{=} \lim_{x\to 0} \frac{2\cos x - 2}{\sin x + x\cdot\cos x} = \left(\frac{0}{0}\right)$

$\overset{H}{=} \lim_{x\to 0} \frac{-2\sin x}{\cos x + \cos x - x\sin x} = \lim_{x\to 0} \frac{-2\sin x}{2\cos x - x\sin x} = \underline{\underline{0}}$

d) $\lim_{x\to 0}(1-\sin x)^{1/x} = (1^{\infty})$

B MODUTAN

1) $\lim_{x\to 0} f(x)^{g(x)} = (1^{\infty}) = e^{\lim_{x\to 0}(f(x)-1)\cdot g(x)}$

$$\lim_{x\to 0}(\cancel{1}-\sin x-\cancel{1})\cdot\frac{1}{x} = \lim_{x\to 0}\frac{-\sin x}{x} = \left(\frac{0}{0}\right) \overset{H}{=}$$

$$\overset{H}{=} \lim_{x\to 0}\frac{-\cos x}{1} = \frac{-\cos 0}{1} = \frac{-1}{1} = -1.$$

Beraz $\lim_{x\to 0}(1-\sin x)^{1/x} = \boxed{e^{-1}}$

2) LOGARITMOEKIN $\ln \lim_{x\to 0} f(x) =$

$$\lim_{x\to 0}\ln f(x) = \lim_{x\to 0}\ln(1-\sin x)^{1/x} =$$

$$= \lim_{x\to 0}\frac{1}{x}\ln(1-\sin x) = \lim_{x\to 0}\frac{\ln(1-\sin x)}{x} = \left(\frac{0}{0}\right) \overset{H}{=}$$

$$\overset{H}{=} \lim_{x\to 0}\frac{\frac{1}{1-\sin x}\cdot(-\cos x)}{1} = \lim_{x\to 0}\frac{-\cos x}{1-\sin x} = \frac{-\cos 0}{1-\sin 0} = -1$$

$$\ln \lim_{x\to 0} f(x) = -1$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(1-\sin x)^{1/x} = \boxed{e^{-1}}$$

g) $\lim_{x \to 0^+} \left(\ln \frac{1}{x}\right)^x = (\infty^0) = \lim_{x \to 0^+} x \cdot$

LOGARITMOAK aplikatuz:

$\ln \lim_{x \to 0^+} \left(\ln \frac{1}{x}\right)^x = \lim_{x \to 0^+} \ln \left(\ln \frac{1}{x}\right)^x = \lim_{x \to 0^+} x \cdot \ln\left(\ln\left(\frac{1}{x}\right)\right) =$

$= 0 \cdot \ln\left(\ln \frac{1}{0}\right) = (0 \cdot \infty) =$

$= \lim_{x \to 0^+} \dfrac{\ln(\ln 1/x)}{1/x} = \left(\frac{\infty}{\infty}\right) \stackrel{H}{=}$

$= \lim_{x \to 0^+} \dfrac{\frac{1}{\ln 1/x} \cdot \frac{1}{1/x} \cdot \cancel{(-x^{-2})}}{\cancel{-x^{-2}}} = \lim_{x \to 0^+} \dfrac{x}{\ln 1/x} = \left(\frac{\infty}{\infty}\right) \stackrel{H}{=}$

$= \lim_{x \to 0^+} \dfrac{1}{\frac{1}{1/x} \cdot (-x^{-2})} = \lim_{x \to 0^+} \dfrac{1}{-x^{-3}} = \lim_{x \to 0^+} x^3 = 0.$

Beraz: $\ln \lim_{x \to 0^+} \left(\ln \frac{1}{x}\right)^x = 0 \Longrightarrow \lim_{x \to 0^+} \left(\ln \frac{1}{x}\right)^x = e^0 = \boxed{1}$

h) $\lim_{x \to -1} (x+1)^{x+1} = (0^0)$

$\ln \lim_{x \to -1} (x+1)^{x+1} = \lim_{x \to -1} \ln (x+1)^{x+1} = \lim_{x \to -1} (x+1) \cdot \ln(x+1) =$

$= (0 \cdot \infty) = \lim_{x \to -1} \dfrac{\ln(x+1)}{1/x+1} = \left(\frac{\infty}{\infty}\right) \stackrel{H}{=}$

$\stackrel{H}{=} \lim_{x \to -1} \dfrac{\frac{1}{x+1}}{-(x+1)^{-2}} = \lim_{x \to -1} \dfrac{(x+1)^{-1}}{-(x+1)^{-2}} = \lim_{x \to -1} -(x+1) = 0.$

Beraz: $\ln \lim_{x \to -1} (x+1)^{x+1} = 0 \longrightarrow \lim_{x \to -1} (x+1)^{x+1} = e^0 = \boxed{1}$